

REAL DOUBLE COSET SPACES AND THEIR INVARIANTS

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ABSTRACT. Let G be a real form of a complex reductive group. Suppose that we are given involutions σ and θ of G . Let $H = G^\sigma$ denote the fixed group of σ and let $K = G^\theta$ denote the fixed group of θ . We are interested in calculating the double coset space $H \backslash G / K$. We use moment map and invariant theoretic techniques to calculate the double cosets, especially the ones that are closed. One salient point of our results is a stratification of a quotient of a compact torus over which the closed double cosets fiber as a collection of trivial bundles.

1. INTRODUCTION

Let U be a compact connected Lie group and let $U_{\mathbb{C}}$ denote its complexification. Assume that we have a real form G of $U_{\mathbb{C}}$ with corresponding real involution φ which preserves U . Assume that we have holomorphic involutions σ and θ of $U_{\mathbb{C}}$, commuting with φ , such that they generate a finite group of automorphisms of the connected center of $U_{\mathbb{C}}$. Let H be an open subgroup of G^σ , the fixed points of σ , and let K denote an open subgroup of G^θ . We are interested in the space of double cosets $H \backslash G / K$. As in [Mat97, §2] one can reduce to the case that

$$(1.0.1) \quad G = HG^0K.$$

For most of this paper we will assume that $K = G^\theta$ and explain later how this assumption can be removed. Then, as in our previous work [HelS01], we identify G/K with a submanifold X of G via the mapping $g \mapsto \beta(g) := g\theta(g^{-1})$. Via this identification, the H action on G/K becomes the $*$ -action on X where $h * x := hx\theta(h)^{-1}$, $h \in H$, $x \in X$. We show that there is a quotient $X // H$ parameterizing the closed orbits (then one can, in principal, determine all orbits). We can assume that the Cartan involution δ of $U_{\mathbb{C}}$ commutes with σ and θ (§2). Let G_0 denote $G \cap U$. Using the results of [HeiS07] we define a kind of moment mapping on X whose zero set \mathcal{M} is $H_0 := (H \cap U)$ -invariant and has the following properties:

- An orbit $H * x$ is closed if and only if it intersects \mathcal{M} .
- For every $x \in X$, the orbit closure $\overline{H * x}$ contains a unique H_0 -orbit in \mathcal{M} .
- The inclusion $\mathcal{M} \rightarrow X$ induces a homeomorphism $\mathcal{M}/H_0 \simeq X // H$.

Now $X_0 := \beta(G_0) \subset G_0$ has the $*$ -action of $G_0 \supset H_0$. Let A be a (connected) torus in X . We say that A is (σ, θ) -split if $\sigma(a) = \theta(a) = a^{-1}$ for all $a \in A$. Now let A_0 be a maximal (σ, θ) -split torus in G_0 (so $A_0 \subset X_0$). Then it follows from [Mat97] (cf. Theorem 4.6) that there is a finite Weyl group W_0^* acting on A_0 such that the inclusion $A_0 \rightarrow X_0$ induces a homeomorphism $A_0/W_0^* \simeq X_0/H_0$. The idea is to try to find a similar result for the H_0 -action on \mathcal{M} .

Let $x \in X$. Then there is a natural submanifold P_x of X which is stable under conjugation by H_x such that $P_x x$ is transversal to the orbit $H * x$ at x . If $H * x$ is closed, then an H_x -stable open subset of $P_x x$ is a slice for the action of H . Moreover, P_x is a symmetric space for the action of H_x . We say that x is a *principal point* if the action of H_x on $\mathcal{S}_x := T_e P_x$ is trivial. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{r}_0$ be the Cartan decomposition of \mathfrak{g} . If $u \in G_0$, then \mathcal{S}_u is δ -stable and decomposes

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into a compact part $\mathcal{S}_u \cap \mathfrak{g}_0$ and a noncompact part $\mathcal{S}_u \cap \mathfrak{r}_0$. There is a natural H_0 -equivariant surjective map $\pi: \mathcal{M} \rightarrow X_0$ where the fiber of π above $u \in X_0$ is $\exp(\mathcal{S}_u \cap \mathfrak{r}_0)u$. Thus \mathcal{M} fibers over X_0 with fiber over u the noncompact part of the transversal at u . We show that there is a natural and finite stratification of A_0 which is W_0^* -stable such that the mapping π is a fiber bundle over each stratum. This in turn implies that $X//H \simeq \mathcal{M}/H_0$ is a fiber bundle over the images of the strata in A_0/W_0^* . If u lies in a stratum S of A_0 , then the fiber over the image of S in A_0/W_0^* is $\exp(\mathcal{S}_u \cap \mathfrak{r}_0)/(H_0)_u$. Moreover, for any maximal σ -split commutative subspace \mathfrak{t} of $\mathcal{S}_u \cap \mathfrak{r}_0$ there is a finite Weyl group $W(S, \mathfrak{t})$ such that $\exp(\mathcal{S} \cap \mathfrak{r}_0)/(H_0)_u \simeq \exp(\mathfrak{t})/W(S, \mathfrak{t})$.

There is another way to parameterize the quotient $X//H$. Let $u \in A_0$ and let A be a δ -stable maximal (σ, θ_u) -split torus. Here θ_u denotes θ followed by conjugation by u . Then $Au \subset \mathcal{M}$. We say that A (or Au) is *standard* if $A \cap U = A \cap A_0$. We say that maximal (σ, θ_{u_i}) -split tori $A_i u_i$, $i = 1, 2$, are *equivalent* if there is an $h \in H$ such that $h * A_1 u_1 = A_2 u_2$. Then we show the following:

- For each stratum of A_0/W_0^* there is at most one associated standard maximal Au . Let $\{A_i u_i\}$ be a maximal collection of pairwise non-equivalent tori coming from the strata. Then $\cup_i A_i u_i \rightarrow X//H$ is surjective. If x is a principal point, then $H * x$ intersects precisely one of the $A_i u_i$ and P_x is a maximal (σ, θ_x) -split torus.
- If $A_1 u_1$ and $A_2 u_2$ are standard maximal, then they are equivalent if and only if there is a $w \in W_0^*$ such that $w * (A_1 u_1 \cap A_0) = A_2 u_2 \cap A_0$.
- If A is a maximal (σ, θ_u) -split torus, then the group of self-equivalences of Au is a finite group. This group acts freely on the set of principal points of Au .

This paper is a natural follow up to [HelS01] where we considered the problem of determining the quotient $G^\sigma \backslash G/G^\theta$ (the complex case, or more generally the case of an algebraically closed field of characteristic not 2). Our techniques were those of invariant theory, i.e., slice theorems, isotropy type stratifications, etc. We also used these techniques here. The new ingredient is the use of moment-map techniques from [HeiS07]. The moment map techniques have also been used in a recent paper of Miebach [Mie07] who works in the setting of Matsuki's characterization of the double coset spaces. The main novelty of our results is the use of stratifications of A_0/W_0^* which help in determining the topological structure of $X//H$. Also, our arguments are somewhat simplified since it is only the action of the group H which is being considered.

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2. CARTAN DECOMPOSITION, KEMPF-NESS SET AND THE QUOTIENT

We now show that one can assume the existence of a Cartan involution commuting with σ and θ .

Proposition 2.1. *Assume that \mathfrak{g} is semisimple. Then there is a Cartan involution δ of \mathfrak{g} such that δ commutes with σ and θ' where θ' differs from θ by conjugation by an inner automorphism of \mathfrak{g} .*

Proof. There are Cartan involutions δ and δ' of \mathfrak{g} which commute with σ and θ , respectively. Moreover, δ and δ' differ by conjugation by an inner automorphism of \mathfrak{g} [Hel78, Ch. III, §7]. \square

Let conjugation by $g \in G$ be denoted by $\text{conj}(g)$.

Corollary 2.2. *For some $g \in G^0$ there is a Cartan involution of G commuting with σ , $\text{conj}(g) \circ \theta \circ \text{conj}(g)^{-1}$ and the involution φ defining $G \subset U_{\mathbb{C}}$.*

Proof. We may write $U_{\mathbb{C}} = (U_{\mathbb{C}}, U_{\mathbb{C}})Z(U_{\mathbb{C}})^0$ where $Z(U_{\mathbb{C}})$ is the center of $U_{\mathbb{C}}$. The two components of the decomposition are preserved by σ , θ and φ . Since $Z(U_{\mathbb{C}})^0$ has a unique maximal compact subgroup T , it is preserved by σ , θ and φ . From Proposition 2.1 we may assume that

we have a Cartan involution δ of $(\mathfrak{g}, \mathfrak{g})$ which commutes with σ and θ . From δ we obtain a connected maximal compact subgroup U' of $(U_{\mathbb{C}}, U_{\mathbb{C}})$. Then $U'T$ is a maximal compact subgroup of $U_{\mathbb{C}}$ which is σ , θ and φ -stable, hence the corresponding Cartan involution of G commutes with the other involutions. \square

Remark 2.3. Changing θ to a conjugate automorphism as above only changes the double coset spaces we consider by an automorphism [Mat97, §1 Remark 2]. Thus from now on we assume that we have a Cartan involution δ commuting with σ , θ and φ and that $(U_{\mathbb{C}})^{\delta} = U$. It is not hard to show that, if σ and θ commute, then one does not need to replace θ by a conjugate to find a suitable δ .

In the following, let G , H and K be as in the introduction, where $K = G^{\theta}$. From θ we have the eigenspace decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Similarly, for σ we have $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$. Let $P := \{x \in G \mid \theta(x) = x^{-1}\}$. Then G acts on P via $*$; $g, x \mapsto g * x := gx\theta(g^{-1})$. We set $\beta(g) = g * e$ for $g \in G$ and we denote $\beta(G)$ by $P_{\theta}(G)$ or X . Then β induces a bijection of G/K onto X . We will now see that X is smooth, and it follows easily that β is a diffeomorphism. Under the diffeomorphism β , the left action of H on G/K becomes the $*$ -action on X .

Lemma 2.4. *Every G -orbit in P is open.*

Proof. Let $x \in P$ and consider the open set of points $\exp(v)x$ where v lies in a small neighborhood of $0 \in \mathfrak{g}$. Then $\exp(v)x$ lies in P if and only if $\exp(\theta(v))\theta(x) = x^{-1}\exp(-v)$ which is equivalent to the condition that $\theta(v) = -(\text{Ad } x^{-1}) \cdot v$. Now let $w = v/2$. Then

$$\exp(w)x\exp(-\theta(w)) = \exp(w)\exp((\text{Ad } x)(-\theta(w)))x = \exp(v/2)\exp(v/2)x = \exp(v)x.$$

Thus $G * x$ contains a neighborhood of x in P . \square

Corollary 2.5. *Every G -orbit in P is closed and is a smooth submanifold. In particular, X is a closed submanifold of G .*

From our Cartan involution δ we have the Cartan decomposition $G = G_0 \exp \mathfrak{t}_0$ where $G_0 = G \cap U$ and $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{t}_0$ is the Cartan decomposition of \mathfrak{g} . The involutions σ and θ preserve G_0 , \mathfrak{g}_0 and \mathfrak{t}_0 . Set $X_0 = P_{\theta}(G_0)$.

Proposition 2.6. *Let G_0 , etc. be as above. Then $X_0 = X \cap U$.*

Proof. Suppose that $x \in X \cap U$, $x = \beta(u\exp(v))$ where $u \in G_0$ and $v \in \mathfrak{t}_0$. Then $x = u * \beta(\exp(v))$. Thus $u^{-1} * x = \beta(\exp(v))$, so that we may assume that $x = \exp(v)\exp(-\theta(v))$ for $v \in \mathfrak{t}_0$. But then $x\exp(\theta(v)) = \exp(v)$ and uniqueness in the Cartan decomposition forces $x = e$. Thus $x \in X_0$. \square

The proof of Lemma 2.4 also gives

Lemma 2.7. *Suppose that $x := u\exp(v) \in X$ where $u \in G_0$ and $v \in \mathfrak{t}_0$. Then $\theta(u) = u^{-1}$ and $\theta(v) = -(\text{Ad } u)v$.*

Corollary 2.8. *There is a G_0 -equivariant projection $\lambda: X \rightarrow X_0$, $u\exp(v) \mapsto u$, where G_0 is acting via $*$.*

Proof. Let $x = u\exp(v) \in X$ as above. Then $u \in P$. By assumption (1.0.1), H_0 acts transitively on the components of X , so we may assume that there is a path from x to e in X , so that there is a path in $P \cap U$ from u to e . Since X_0 contains the component of $P \cap U$ containing e we have $u \in X_0$. If $u' \in G_0$, then $u' * x = (u' * u)\exp(\text{Ad } \theta(u')v) \in X$ where $\text{Ad } \theta(u')v \in \mathfrak{t}_0$. Thus λ is equivariant. \square

Remark 2.9. There is another way to view $\lambda: X \rightarrow X_0$. From [HeiS07, Theorem 9.3] there is an isomorphism $G_0 \times^{K_0} (\mathfrak{p} \cap \mathfrak{t}_0) \simeq G/K$, $[u, \xi] \mapsto u\exp(\xi)$, where $K_0 = K \cap U$. This gives us a G_0 -equivariant map $G/K \rightarrow G_0/K_0$. Applying β one gets the mapping λ .

2.10. Conditions on σ and θ . Our assumption on σ and θ is that they generate a finite group of automorphisms of $Z(U)^0$. This assumption is related to that of Matsuki [Mat97, §2]:

Proposition 2.11. *The following are equivalent.*

- (1) *The subgroup of $\text{Aut}(Z(U)^0)$ generated by σ and θ is finite.*
- (2) *The image of $\sigma\theta$ in $\text{Aut}(Z(U)^0) \simeq \text{GL}(n, \mathbb{Z})$ is semisimple with eigenvalues of norm 1.*

Proof. Clearly (1) implies (2). Now assume (2). The image of $\sigma\theta$ in $\text{Aut}(Z(U)^0) \simeq \text{GL}(n, \mathbb{Z})$ has eigenvalues that satisfy a monic polynomial equation with integer coefficients. Since the eigenvalues have norm 1, they must be roots of unity. Since $\sigma\theta$ is semisimple, it follows that $\sigma\theta$ has finite order. Thus σ and θ generate a finite subgroup of $\text{Aut}(Z(U)^0)$. \square

Lemma 2.12. *There is an inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{u} which is U , σ and θ -invariant.*

Proof. Write $U = U_s Z(U)^0$ where $U_s = (U, U)$ is the semisimple part of U . Since $\text{Aut}(U_s)$ is a finite extension of the inner automorphisms $\text{Int } U_s$ of U_s , we find that σ , θ and U_s generate a compact group of automorphisms of \mathfrak{u}_s , so that we can find an inner product invariant under this compact group. Since σ and θ generate a finite group of automorphisms of $\mathfrak{z}(\mathfrak{u})$, it has an invariant inner product. Putting the two inner products together gives the desired result. \square

2.13. Moment mapping and Kempf-Ness set. We have the Cartan decomposition $U_{\mathbb{C}} \simeq U \times i\mathfrak{u} \simeq U \times \exp(i\mathfrak{u})$. Let $\langle \cdot, \cdot \rangle$ be an inner product on \mathfrak{u} which is U , θ and σ -invariant. We also consider $\langle \cdot, \cdot \rangle$ to be an inner product on \mathfrak{u}^* . The function $\rho: U_{\mathbb{C}} \rightarrow \mathbb{R}$, $\rho(u \exp(i\eta)) := (1/2)\langle \eta, \eta \rangle$, $u \in U$, $\eta \in \mathfrak{u}$, is a strictly plurisubharmonic exhaustion of $U_{\mathbb{C}}$ (see [HeiS07, §9]) and it is invariant under left and right multiplication by elements of U . Associated to ρ we have a Kähler form $\omega = 2i\partial\bar{\partial}\rho$ and a moment mapping $\nu: U_{\mathbb{C}} \rightarrow \mathfrak{u}^*$. If $g \in U_{\mathbb{C}}$ and $\xi \in \mathfrak{u}$, then $\nu(g)(\xi) = \frac{d}{dt}|_{t=0}\rho(g \exp(-it\xi))$. Note that ν is U -equivariant, where U is acting by right multiplication on $U_{\mathbb{C}}$ and $u \in U$ acts on \mathfrak{u}^* by $\alpha \mapsto \alpha \circ \text{Ad}(u^{-1})$. Let ν^ξ denote ν followed by evaluation at $\xi \in \mathfrak{u}$. A computation [HeiS07, Lemma 9.1] shows that ν has the following simple form:

$$(2.13.1) \quad \nu^\xi(u \exp(i\eta)) = \langle \xi, \eta \rangle, \quad u \in U; \quad \xi, \eta \in \mathfrak{u}.$$

Thus ν is, essentially, projection of $U_{\mathbb{C}} \simeq U \times i\mathfrak{u} \simeq U \times \mathfrak{u}^*$ onto the second factor. Now we consider the action of U on $U_{\mathbb{C}}$ given by $u * g = ug\theta(u)^{-1}$. Note that ρ is still U -invariant.

Lemma 2.14. *Let U act on $U_{\mathbb{C}}$ by $*$ and let $\mu: U_{\mathbb{C}} \rightarrow \mathfrak{u}^*$ denote the corresponding moment mapping; $\mu^\xi(g) = \frac{d}{dt}|_{t=0}\rho(\exp(it\xi)g \exp(-it\theta(\xi)))$. Then*

$$\mu^\xi(u \exp(i\eta)) = \langle \eta, \theta(\xi) - \text{Ad}(u^{-1})(\xi) \rangle, \quad u \in U; \quad \eta, \xi \in \mathfrak{u}.$$

Proof. The product rule shows that

$$\mu^\xi(u \exp(i\eta)) = \frac{d}{dt}|_{t=0}\rho(\exp(it\xi)u \exp(i\eta)) + \frac{d}{dt}|_{t=0}\rho(u \exp(i\eta) \exp(-it\theta(\xi)))$$

where by (2.13.1) the second term is $\langle \eta, \theta(\xi) \rangle$. Now

$$\rho(\exp(it\xi)u \exp(i\eta)) = \rho(u \exp(it \text{Ad}(u^{-1})\xi) \exp(i\eta)) = \rho(\exp(it \text{Ad}(u^{-1})\xi) \exp(i\eta)).$$

Let ζ denote $\text{Ad}(u^{-1})\xi$. Write $\exp(i\eta) \exp(-it\zeta) = u(t) \exp(i\gamma(t))$ where $u(t) \in U$ and $\gamma(t) \in \mathfrak{u}$. Then $\rho(\exp(i\eta) \exp(-it\zeta)) = 1/2\langle \gamma(t), \gamma(t) \rangle$ and by (2.13.1), $\frac{d}{dt}|_{t=0}\langle \gamma(t), \gamma(t) \rangle = \langle \eta, \zeta \rangle$. It follows that

$$\rho(\exp(it\zeta) \exp(-i\eta)) = \rho(\exp(-i\gamma(t))u(t)^{-1}) = 1/2\langle \gamma(t), \gamma(t) \rangle$$

and that

$$\frac{d}{dt}|_{t=0}\rho(\exp(it\xi)u \exp(i\eta)) = \langle -\eta, \zeta \rangle = \langle \eta, -\text{Ad}(u^{-1})\xi \rangle.$$

□

Let $x := u \exp(i\eta) \in X$ where $u \in G_0$ and $\eta \in i\mathfrak{r}_0$. By restriction, the moment mapping μ gives us a mapping $X \rightarrow \mathfrak{u}^* \rightarrow (i\mathfrak{h} \cap \mathfrak{u})^*$, which we also denote by μ . Then μ is the relevant moment mapping when one considers the action of H on X (see [HeiS07, §5] and §2.15 below). Let \mathcal{M} denote the Kempf-Ness set, i.e., $\mathcal{M} = \{x \in X \mid \mu(x) = 0\}$. From Lemma 2.14 we see that

$$\mathcal{M} \subset \{u \exp(i\eta) \mid \theta(\eta) - \text{Ad}(u)\eta \perp i\mathfrak{h} \cap \mathfrak{u}\}.$$

From Lemma 2.7 and Corollary 2.8 we have $u \in X_0$ and $\theta(\eta) = (-\text{Ad } u)(\eta)$ so that $2\theta(\eta) \perp i\mathfrak{h} \cap \mathfrak{u}$. Now $i\mathfrak{g} = i\mathfrak{h} \oplus i\mathfrak{q}$ so that the perpendicular to $i\mathfrak{h} \cap \mathfrak{u}$ in $i\mathfrak{g} \cap \mathfrak{u} = i\mathfrak{r}_0$ is $i\mathfrak{q} \cap \mathfrak{u}$. Thus when $u \exp(i\eta) \in \mathcal{M}$ we have that $\theta(\eta) \in i\mathfrak{q} \cap i\mathfrak{r}_0$. We finally get that

$$(2.14.1) \quad \mathcal{M} = \{u \exp(\zeta) \mid u \in X_0, \theta(\zeta) \in \mathfrak{q} \cap \mathfrak{r}_0 \text{ and } \theta(\zeta) = -\text{Ad}(u)(\zeta)\}.$$

2.15. The quotient. Recall that $H_0 = H \cap U$. Since the moment mapping comes from a strictly plurisubharmonic exhaustion, from [HeiS07, 10.2, 11.2, 11.15, 13.4] we have the following result

Theorem 2.16. *Let $x \in X$. Then the orbit closure $\overline{H * x}$ contains a point x_0 of \mathcal{M} . Moreover, $\overline{H * x} \cap \mathcal{M} = H_0 x_0$ is a single H_0 -orbit.*

We now define an equivalence relation on X by $x \sim y$ if the corresponding H_0 -orbits in \mathcal{M} coincide. We define the quotient $X//H$ to be the set of all equivalence classes with the quotient topology. From [HeiS07, 11.2, 13.5] we have

Theorem 2.17. *The inclusion $\mathcal{M} \rightarrow X$ induces a homeomorphism $\mathcal{M}/H_0 \simeq X//H$. The closed H -orbits in X are precisely those which intersect \mathcal{M} .*

3. TRANSVERSALS AND THE SLICE THEOREM

Let $x \in X$. Let θ_x denote $\text{conj}(x) \circ \theta$ and let τ_x denote $\theta_x \sigma$. Then θ_x is an involution of G . Let $G^{(x)}$ denote the fixed points of τ_x . Then $\text{conj}(x^{-1})$ and $\tau = \theta \sigma$ are the same on $G^{(x)}$ as are σ and θ_x . Moreover, $G^{(x)}$ is σ and θ_x -stable. Note that $G^{(x)}$ is not necessarily reductive (see Corollary 5.3 below). Let \mathcal{S}_x denote $\{Z \in \mathfrak{g} \mid \theta_x(Z) = \sigma(Z) = -Z\}$. Let P_x denote the component of $P_\theta(G^{(x)})$ containing e . Let $\mathfrak{p}^{(x)}$ denote $\{Z \in \mathfrak{g} \mid \theta_x(Z) = -Z\}$ and let $\mathfrak{k}^{(x)}$ denote $\{Z \in \mathfrak{g} \mid \theta_x(Z) = Z\}$. Then $\mathfrak{g} = \mathfrak{k}^{(x)} \oplus \mathfrak{p}^{(x)}$. The subgroup of $G^{(x)}$ fixed by σ is just H_x , the isotropy group of H at x .

We say that a locally closed H_x -stable smooth subset Q_x of X is a *transversal* at x if $T_x(Q_x)$ is a complement to $T_x(H * x)$ in $T_x X$. The following theorem generalizes the results in [HelS01].

Theorem 3.1. *Let $x \in X$. Then P_x is a transversal to $H * x$. Moreover, $T_e(P_x) = \mathcal{S}_x$ and $T_e(H_x) = \mathfrak{k}^{(x)} \cap \mathfrak{h}$.*

Proof. We have the smooth map $\psi: G \rightarrow Xx^{-1}$, $g \mapsto (g * x)x^{-1}$. The differential $d\psi$ of ψ at e sends $Z \in \mathfrak{g}$ to $Z - \theta_x Z$. The square of $d\psi$ is $2d\psi$, so that $d\psi$ is (up to a constant) a projection of \mathfrak{g} onto $T_e(Xx^{-1})$. Now $d\psi(\mathfrak{h})$ is the tangent space to $(H * x)x^{-1}$ at e . The kernel of $d\psi$ is $\mathfrak{k}^{(x)}$, so that $\mathfrak{p}^{(x)}$ is complementary to the kernel of $d\psi$. In fact, $d\psi$ is multiplication by 2 on $\mathfrak{p}^{(x)}$. Thus the Lie algebra of H_x is $\mathfrak{h} \cap \mathfrak{k}^{(x)}$ and the complement to $T_e((H * x)x^{-1})$ in $T_e(Xx^{-1})$ is $d\psi(\mathfrak{q} \cap \mathfrak{p}^{(x)}) = \mathfrak{q} \cap \mathfrak{p}^{(x)} = \mathcal{S}_x$. Now $T_e P_x = \{Z \in \mathfrak{g}^{(x)} \mid \sigma(Z) = -Z\} = \{Z \in \mathfrak{q} \mid \theta_x(Z) = -Z\} = \mathcal{S}_x$. It follows that P_x is a transversal to $H * x$ at x . □

From [HeiS07, 14.10] (see also [HelS01, 2.7]) we obtain the slice theorem.

Corollary 3.2. *Suppose that $H * x$ is closed. Then there is an open H_x -stable subset $S \ni x$ of P_x which is a slice at x for the action of H on X . In other words, the canonical mapping $H \times^{H_x} S \rightarrow X$, $[h, s] \mapsto h * s$, is an H -equivariant diffeomorphism onto an open subset of X .*

We now compare the transversals along H -orbits.

Proposition 3.3. *Let $x \in X$ and let $h \in H$. Then $\text{conj}(h)(G^{(x)}) = G^{(h*x)}$ and $\text{conj}(h)(P_x) = P_{h*x}$. Thus $h*(P_x x) = \text{conj}(h)(P_x)h*x = P_{h*x}h*x$, hence h carries the transversal at x to the transversal at $h*x$.*

Proof. Let $g \in G^{(x)}$, i.e., assume that $\tau_x(g) = g$. Then

$$\tau_{h*x}(\text{conj}(h)g) = hx\theta(h)^{-1}\theta(h)\tau(g)\theta(h)^{-1}\theta(h)x^{-1}h^{-1} = hx\tau(g)x^{-1}h^{-1} = \text{conj}(h)g.$$

Thus $\text{conj}(h)G^{(x)} \subset G^{(h*x)}$ and similarly $\text{conj}(h^{-1})G^{(h*x)} \subset G^{(x)}$, so we have equality. It follows that $\{g\sigma(g)^{-1} \mid g \in G^{(x)}\}$ is sent by $\text{conj}(h)$ onto $\{g\sigma(g^{-1}) \mid g \in G^{(h*x)}\}$, so h maps $P_x x$ isomorphically onto $P_{h*x}(h*x)$. \square

By a torus in $X \subset G$ we mean a connected commutative group of semisimple elements. Thus \mathbb{R}^* is not a torus for us, but its identity component is. Now as above, we have the following

Proposition 3.4. *Let $x \in X$, let $h \in H$ and let A be a (σ, θ_x) -split torus in G . Then $\text{conj}(h)A$ is (σ, θ_{h*x}) -split and $h*(Ax) = (\text{conj}(h)A)h*x$.*

Definition 3.5. We say that $x \in X$ is a *principal point* if $H*x$ is closed and the slice representation at x is trivial, i.e., H_x acts trivially on \mathcal{S}_x . We say that $x \in \mathcal{M}$ is *principal* if it is principal in X .

Remark 3.6. If x is principal, then the slice representation at x must have dimension $\dim X//H$ which is the same as the dimension of a maximal (σ, θ) -split torus in X . But this is the same as the dimension of a maximal (σ, θ_x) -split torus in P_x , hence P_x must be a maximal (σ, θ_x) -split torus. Finally, it is well-known that the principal points for the H_x -action on P_x are open and dense, hence the same is true for H acting on X .

4. THE COMPACT CASE

We first need to compute the quotient of X_0 by the action of H_0 . This is in [Mat97, §3], but we need Corollary 4.5. A variant of an argument in [HelS01, 6.7] gives this result.

Let A_0 be a fixed maximal (σ, θ) -split torus in X_0 . Set $W_0^* = N_0^*/Z_0^*$ where $N_0^* = \{h \in H_0 \mid h*A_0 = A_0\}$ and $Z_0^* = \{h \in H_0 \mid h*a = a \text{ for all } a \in A_0\}$. We have the following elementary lemma (c.f. [HelS01, 1.10, 4.5])

Lemma 4.1. (1) $N_0^* = \{h \in N_{H_0}(A_0) \mid \beta(h) \in A_0\}$.

(2) For $h \in H_0$, $\beta(h) \in A_0$ if and only if there is an $s \in A_0$ such that $s^{-1}h \in K_0$, in which case $s^2 = \beta(h)$.

(3) For $h \in H_0$, $h \in N_0^*$ if and only if $hA_0K_0 = A_0K_0$ and $h \in Z_0^*$ if and only if $haK_0 = aK_0$ for all $a \in A_0$.

(4) If $h \in N_0^*$, let $s \in A_0$ such that $hK_0 = sK_0$. Then $haK_0 = (h*a)s^{-1}K_0 = hah^{-1}sK_0$ for all $a \in A_0$.

Remark 4.2. It follows from Theorem 3.1 that the intersection of H_0*e and A_0 is discrete, hence finite. Thus for any $h \in N_0^*$, there are only finitely many possibilities for $\beta(h)$. Hence W_0^* is finite since $N_{H_0}(A_0)/Z_{H_0}(A_0)$ is finite.

Proposition 4.3. *Let $h \in H_0$ such that $h*Y = Y'$ where $Y, Y' \subset A_0$ are nonempty. Then there is a $w \in W_0^*$ such that $w*y = h*y$ for all $y \in Y$.*

Proof. Let $G_0^Y = \{g \in G_0 \mid \tau_y(g) = g \text{ for all } y \in Y\}$ and define $G_0^{Y'}$ similarly. Then a calculation shows that $\text{conj}(h)G_0^Y = G_0^{Y'}$. Now A_0 is maximal σ -split in both G_0^Y and $G_0^{Y'}$, and $\text{conj}(h^{-1})A_0$ is maximal σ -split in G_0^Y . Thus there is an element g in the identity component of $(G_0^Y)^\sigma$ such

that $g^{-1}h^{-1}A_0 = A_0$. Since $g \in G_0^Y$ and $\sigma(g) = g$, it follows that for all $y \in Y$ we have $\theta_y(g) = g$ and hence $g * y = y$. Thus $hg \in N_{H_0}(A_0)$ and hg acts on Y in the same way as h . Since $hg * y \in A_0$ and $\text{conj}(hg)y \in A_0$ for $y \in Y$, it follows that $hg \in N_0^*$. Hence hg gives the requisite element of W_0^* . \square

Now we consider the case where K is replaced by an open subgroup. Let K' denote an open subgroup of K , let K'_0 denote $K' \cap U$ and let X'_0 denote G_0/K'_0 . We have the H_0 -equivariant covering map $\beta: X'_0 \rightarrow X_0$. Let $N'_0 = \{h \in N_{H_0}(A_0) \mid hA_0K'_0 = A_0K'_0\}$, let $Z'_0 = \{h \in H_0 \mid hA_0K'_0 = A_0K'_0 \text{ for all } a \in A_0\}$ and define $W'_0 = N'_0/Z'_0$. Let $A_0^{(2)}$ denote the elements of A_0 of order 2. Then $A_0^{(2)} = A_0 \cap K$. There is a natural morphism $W'_0 \rightarrow W_0^*$ and we have

Lemma 4.4. *The canonical mapping $W'_0 \rightarrow W_0^*$ has kernel $A_0^{(2)}K'_0 \cap Z_{H_0}(A_0)K'_0$. The image is represented by $\{h \in N_0^* \mid s^{-1}h \in K'_0 \text{ for some } s \in A_0\}$.*

Corollary 4.5. *Let Y, Y' be nonempty subsets of A_0 and suppose that $h \in H_0$ such that $hY(K'_0) = Y'(K'_0)$. Then there is a $w \in W'_0$ such that $hy(K'_0) = wy(K'_0)$ for all $y \in Y$.*

Proof. As in the proof of Proposition 4.3 we can find g in the identity component of $M := (G^Y)^\sigma$ such that $hg \in N_0^*$. We saw that $YK_0 \subset X_0^M$, hence the elements of the Lie algebra \mathfrak{m} , considered as vector fields on X_0 , vanish on YK_0 . It follows that the elements of \mathfrak{m} , considered as vector fields on X'_0 , vanish on $Y(K'_0)$. Thus g fixes $Y(K'_0)$. Hence we may reduce to the case that $h \in N_0^*$. For $y \in Y$ we have $hy(K'_0) = y'(K'_0)$ where $y' \in A_0$. It follows that $(y')^{-1}hyh^{-1} \in K'_0$ where $s := (y')^{-1}hyh^{-1} \in A_0$. Thus h induces an element of W'_0 . \square

From [Mat97, §3] we have

Theorem 4.6. (1) $G_0 = H_0A_0K'_0$.

(2) *The inclusion $A_0K'_0 \subset X'_0$ induces a homeomorphism $(A_0K'_0)/W'_0 \simeq X'_0/H_0$.*

Note that Corollary 4.5 is a strengthening of (2) above.

5. PARAMETERIZATION OF THE QUOTIENT

We find it useful to consider the Kempf-Ness set \mathcal{M} with the orders of the compact and noncompact parts reversed. Using (2.14.1) one easily shows

Proposition 5.1. *We have*

$$\mathcal{M} = \{\exp(\xi)u \mid u \in X_0, \xi \in \mathcal{S}_u \cap \mathfrak{r}_0\}.$$

Remark 5.2. The description of \mathcal{M} shows that \mathcal{M} fibers over X_0 with fiber over $u \in X_0$ the noncompact part $\exp(\mathcal{S}_u \cap \mathfrak{r}_0)u$ of the transversal P_uu .

Corollary 5.3. *Let $x \in X$ such that $H * x$ is closed. Then $\tau_x = \theta_x\sigma$ is semisimple so that $G^{(x)}$ is reductive.*

Proof. First assume that $x \in \mathcal{M}$, so that $x = \exp(\xi)u$ for $u \in X_0$ and $\xi \in \mathcal{S}_u \cap \mathfrak{r}_0$. It follows from Lemma 2.12 that τ_u is semisimple, and since $\xi \in \mathcal{S}_u$, $\tau_u(\xi) = \xi$. Since $i\xi \in \mathfrak{u}$, $\text{ad } i\xi$ and $\text{ad } \xi$ are semisimple endomorphisms of $\mathfrak{u}_{\mathbb{C}}$, hence $\text{conj}(\exp(\xi))$ is a semisimple automorphism of $U_{\mathbb{C}}$ commuting with τ_u . Thus $\tau_x = \text{conj}(\exp(\xi))\tau_u$ is semisimple.

Now suppose that $h \in H$ and $x \in \mathcal{M}$. Then one computes that $\tau_{h*x} = \text{conj}(h)\tau_x\text{conj}(h^{-1})$, so τ_{h*x} is semisimple. \square

Lemma 5.4. *Let $u \in A_0$ and let A be a δ -stable (σ, θ_u) -split torus. Then $Au \subset \mathcal{M}$.*

Proof. We may write $A = BC$ where B is a δ -split torus and C is δ -fixed, both tori being σ -split. Let $b = \exp(Z) \in B$ and $c \in C$. Since bc is θ_u -split, it follows that both b and c are θ_u -split, so that Z is θ_{cu} -split. Hence $Z \in \mathcal{S}_{cu} \cap \mathfrak{r}_0$ and $bcu \in \mathcal{M}$. \square

Remark 5.5. If $Z \in \mathfrak{r}_0$, then $\text{ad } Z$ acts on \mathfrak{g} with real eigenvalues, hence an element of \mathfrak{g} is fixed by $\text{Ad}(\exp(Z))$ if and only if it is annihilated by $\text{ad } Z$.

We now give a small subset of \mathcal{M} mapping onto the quotient $X//H$.

Theorem 5.6. *There are points $u_1, \dots, u_s \in A_0$ and maximal (σ, θ_{u_i}) -split δ -stable tori A_i such that every closed H -orbit in X intersects $\cup A_i u_i$. Moreover, every principal orbit intersects exactly one of the $A_i u_i$.*

Proof. Let $x \in \mathcal{M}$, so that $x = \exp(Z)u$ where $u \in X_0$ and $Z \in \mathcal{S}_u \cap \mathfrak{r}_0$. Theorem 4.6 allows us to assume that $u \in A_0$. Now the transversal at u is $P_u u$ where P_u is the symmetric space associated to $G^{(u)}$ and σ . We have the transversal $Q_x x$ at x for the action of H_u on $P_u u$. Then

$$\begin{aligned} T_e(Q_x) &= \{Y \in \mathfrak{g}^{(u)} \mid \sigma(Y) = \theta_x(Y) = -Y\} \\ &= \{Y \in \mathcal{S}_u \mid \text{Ad}(\exp(Z))Y = Y\} = \{Y \in \mathcal{S}_u \mid [Z, Y] = 0\}. \end{aligned}$$

If x is principal, then $\exp(Z)$ is principal for the action of H_u on P_u , hence Q_x has to be a maximal (σ, θ_u) -split torus A in G . By construction, A is δ -stable, hence $Au \subset \mathcal{M}$. Now we have $\exp(Z) \in A$ and $x \in Au$. We have shown that the principal orbits intersect a union $\cup A_i u_i$, but we do not yet know that finitely many $A_i u_i$ suffice.

Now suppose that we have $A_1 u_1$ and $A_2 u_2$ where $u_1, u_2 \in A_0$ and the A_i are δ -stable maximal (σ, θ_{u_i}) -split tori in X , $i = 1, 2$. Suppose that there are principal points $x_i := a_i u_i$ and an $h \in H_0$ such that $h * x_1 = x_2$. Since x_1 is principal, P_{x_1} is a maximal (σ, θ_{x_1}) -split torus. But A_1 is σ -split and θ_{x_1} -split (since $a_1 \in A_1$), so $P_{x_1} = A_1$ and for the same reason, $P_{x_2} = A_2$. Then Proposition 3.4 shows that $\text{conj}(h)A_1 = A_2$ and hence $h * A_1 u_1 = A_2 u_2$. Thus the principal points of \mathcal{M}/H_0 are the image of the disjoint union of the principal points of sets $A_i u_i$. Moreover, the principal points of the $A_i u_i$ have open image in \mathcal{M}/H_0 , so that each irreducible component of the principal points of \mathcal{M}/H_0 lies in the image of a single $A_i u_i$. Now the principal points of \mathcal{M}/H_0 are a semialgebraic subset of \mathcal{M}/H_0 , so that there are finitely many components. Hence we only need a finite number of $A_i u_i$. Since the set of principal orbits is open and dense in X , the same is true for \mathcal{M} . It follows that $H_0 * \cup_i A_i u_i = \mathcal{M}$. Hence every closed H -orbit in X contains a point of $\cup_i A_i u_i$. \square

Definition 5.7. Let A_i be maximal (σ, θ_{u_i}) -split tori where $u_i \in A_0$, $i = 1, 2$. We say that $A_1 u_1$ and $A_2 u_2$ are *equivalent* if there is an $h \in H$ such that $h * A_1 u_1 = A_2 u_2$. Equivalently, $\text{conj}(h)A_1 = A_2$ and $h * u_1 \in A_2 u_2$. We define the Weyl group $W_H^*(A_1 u_1)$ to be $N_H^*(A_1 u_1)/Z_H^*(A_1 u_1)$ where the $*$ just reinforces the fact that H is acting via $*$ and not by conjugation.

Remark 5.8. If $u = e$, then $W_H^*(Au) = W_H^*(A)$ is the usual (twisted) Weyl group of A .

We saw above that whenever $A_1 u_1$ and $A_2 u_2$ have images in $X//H$ with a common principal point, then they are equivalent by an element of H_0 . The self equivalences of $A_1 u_1$ are just $W_H^*(A_1 u_1)$.

Example 5.9. Here $U_{\mathbb{C}} = \text{SL}(2, \mathbb{C})$, $U = \text{SU}(2, \mathbb{C})$ and $G = \text{SL}_2(\mathbb{R})$. We have the involution θ which is conjugation with $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and the involution σ which is conjugation with $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $\tau = \theta\sigma$ is conjugation with $v := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\sigma\theta$ is conjugation with $-v$, so that σ and θ commute. The group H is $\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbb{R}^* \}$, $\mathfrak{q} = \{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} : b, c \in \mathbb{R} \}$, $K = \{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{R}, a^2 - b^2 = 1 \}$ and $\mathfrak{p} = \{ \begin{pmatrix} a & b \\ -b & -a \end{pmatrix} \mid a, b \in \mathbb{R} \}$. The Cartan involution δ of $U_{\mathbb{C}}$ is conjugate inverse transpose,

δ commutes with σ and θ and $\mathfrak{r}_0 = \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$. We have $G_0 = G \cap U = \mathrm{SO}(2, \mathbb{R}) = A_0 = X_0$ and $X := P_\theta(G) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid ad + b^2 = 1 \right\}$. The element $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in H$ sends $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ to $\begin{pmatrix} \lambda^2 a & b \\ -b & \lambda^{-2} a \end{pmatrix}$. Thus $H_0 = \{\pm I\}$ acts trivially on X so that $X//H \simeq \mathcal{M}$ and the orbits of H are connected.

Use coordinates $x = b$, $y = (a + d)/2$ and $z = (a - d)/2$ on X . In these coordinates, X is just the hyperboloid in 3-space given by the equation $x^2 + y^2 = 1 + z^2$. The action of H fixes x and on y and z it is given by matrices $\left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a^2 - b^2 = 1, a > 0 \right\}$. Now consider the intersection $X \cap \{x = c\}$ for c fixed.

- (1) If $c^2 \neq 1$, then $X \cap \{x = c\}$ is the hyperbola $y^2 - z^2 = 1 - c^2$, each of whose branches is an H -orbit. If $1 - c^2 > 0$ (resp. $1 - c^2 < 0$), then the H -orbit contains a unique point where $z = 0$ (resp. $y = 0$).
- (2) If $c^2 = 1$, then $X \cap \{x = c\}$ is the union of the two lines $\{(x, y, z) = (c, s, s)\}$ and $\{(x, y, z) = (c, s, -s)\}$ where $s \in \mathbb{R}$. There are two H -fixed points $(\pm 1, 0, 0)$ and eight non-closed H -orbits $\{(\pm 1, \pm s, \pm s) \mid s > 0\}$.

Set $M := \{(x, y, z) \in X \mid y = 0 \text{ or } z = 0\}$ which is the union of $A_0 = X_0$ and the hyperbola $\{x^2 - z^2 = 1, y = 0\}$. From the above, each closed H -orbit intersects M in a unique point. We show that $M = \mathcal{M}$.

Let $u \in A_0$. From Remark 5.2 the fiber of \mathcal{M} over u is $\exp(\mathcal{S}_u \cap \mathfrak{r}_0)u$. If $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \in \mathfrak{q}$ lies in \mathcal{S}_u , then it is fixed by τ_u which is conjugation with uv . For $u \neq \pm v$, conjugation by uv fixes only the Lie algebra $\left\{ \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \right\}$ of A_0 . Thus $\mathcal{S}_u \cap \mathfrak{r}_0 = \{0\}$ and the fiber of \mathcal{M} over u is just the point u . Now suppose that $u = \pm v$. Then uv acts trivially on \mathfrak{q} , so that $\mathcal{S}_u \cap \mathfrak{r}_0 = \left\{ \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} \right\}$ whose exponential is the torus $A := K^0 = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a^2 - b^2 = 1 \text{ and } a > 0 \right\}$. Thus the fiber of \mathcal{M} above each point $\pm v$ is the translated torus $A(\pm v)$ which, viewed in our coordinates on X , is the branch of $\{x^2 - z^2 = 1, y = 0\}$ through v . Hence $M = \mathcal{M}$. Note that the principal points in \mathcal{M} are just the complement of $\pm v$. The (translated) tori of Theorem 5.6 are A_0 , Av and $A(-v)$, each with trivial Weyl group.

5.10. Non-closed orbits. It is possible to classify all the orbits of H on X , not just the closed orbits. Let $H * y$ be a non-closed orbit, and let x be a closed orbit in the closure of $H * y$ (see Theorems 2.16 and 2.17). We may assume that $x \in \mathcal{M}$. Then $H * y$ has to intersect the transversal $P_x x$, so we may assume that $y \in P_x x$. Recall that $G^{(x)}$ is reductive (Corollary 5.3). Now it is well-known that the non-closed orbits for the action of H_x on P_x are the nontrivial unipotent orbits, i.e., those consisting of unipotent elements of $G^{(x)}$ [Ric82]. But the unipotent elements are those of the form $\exp(n)$ where $n \in \mathcal{S}_x$ is a nilpotent element of the semisimple part of $\mathfrak{g}^{(x)}$. There are only finitely many H_x -conjugacy classes of such nilpotent elements, so one is reduced to a computation involving symmetric space representations. Hence one can compute all the H -orbits on X . In Example 5.9 above, the only points $x \in \mathcal{M}$ where the slice representation of H_x is nontrivial are the non-principal points v and $-v$. At v , we have $G^{(v)} = G$ and $\theta_v = \sigma$. Thus $\mathcal{S}_v = \left\{ \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$. The representation of $H_v = H \simeq \mathbb{R}^*$ on \mathcal{S}_v has weights 2 and -2 of multiplicity one, and the corresponding weight spaces are generated by $n_2 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $n_{-2} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, respectively. There are two nontrivial H -orbits in each of $\mathbb{R} \cdot n_{\pm 2}$. Thus there are four non-closed orbits in X with closure containing v , as we saw above. Similarly, there are four non-closed orbits whose closures contain $-v$.

6. SOME RESULTS ON LIE ALGEBRAS

In this section only, G will denote a general real reductive Lie group with Cartan involution δ commuting with an involution σ . We have the Cartan decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{r}_0$ where $G_0 = G^\delta$ and we have the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ relative to σ . The groups H and H_0 are defined as before.

Proposition 6.1. *Let $\mathfrak{b}_0 \subset \mathfrak{g}_0$ be a maximal σ -split commutative subalgebra. Let $\mathfrak{b}_1 \subset \mathfrak{r}_0 \cap \mathfrak{q}$ be maximal abelian in the centralizer of \mathfrak{b}_0 . Then*

- (1) $\mathfrak{b} := \mathfrak{b}_0 \oplus \mathfrak{b}_1$ is maximal abelian in \mathfrak{q} .
- (2) Let \mathfrak{b}'_1 be another choice of \mathfrak{b}_1 . Then \mathfrak{b}_1 and \mathfrak{b}'_1 are conjugate by an element of the connected centralizer of \mathfrak{b}_0 in H_0 .

Proof. Part (1) is obvious. For (2) replace G by the connected centralizer of \mathfrak{b}_0 and then divide by the center. This reduces us to the case that $\mathfrak{g}_0 \cap \mathfrak{q} = \{0\}$, so that $\mathfrak{g}_0 \subset \mathfrak{h}$ and $G_0 = H_0$. Moreover, \mathfrak{b}_1 and \mathfrak{b}'_1 are maximal abelian subspaces of $\mathfrak{q} \subset \mathfrak{r}_0$. Now \mathfrak{b}_1 and \mathfrak{b}'_1 can be extended to maximal abelian subspaces of \mathfrak{r}_0 . But all such subspaces are $(G_0)^0$ -conjugate. Since $G_0 = H_0$ preserves \mathfrak{q} , we see that \mathfrak{b}_1 and \mathfrak{b}'_1 are $(H_0)^0$ -conjugate. \square

Corollary 6.2. (1) *Let \mathfrak{b}_1 be as above and set $B_0 = \exp(\mathfrak{b}_0)$. Then $B := B_0 \exp(\mathfrak{b}_1)$ is a maximal σ -split torus in G .*

- (2) *Let B' be a maximal σ -split torus in G such that the maximal compact subgroup of B' has the same dimension as B_0 . Then B' is $(H_0)^0$ -conjugate to B .*

Proof. Part (1) is immediate from Proposition 6.1. As for (2), we may assume that B' is δ -stable [OM80, Remark after Lemma 5]. Then we may decompose B' as $(B' \cap U) \times (B' \cap R_0)$ where $R_0 = \exp(\mathfrak{r}_0)$. Since $B' \cap U$ has the same dimension as B_0 , we know that $B' \cap U$ and B_0 are $(H_0)^0$ -conjugate. Thus we may assume that $B' \cap U = B_0$. By Proposition 6.1 there is an element of $(H_0)^0$ which centralizes B_0 and conjugates $B' \cap R_0$ into $B \cap R_0$. \square

If \mathfrak{m} and \mathfrak{n} are subalgebras of \mathfrak{g} , we denote by $\mathfrak{m}^{\mathfrak{n}}$ the centralizer of \mathfrak{n} in \mathfrak{m} .

Lemma 6.3. *The subalgebra $\mathfrak{q}^{\mathfrak{h}}$ lies in the center of \mathfrak{g} .*

Proof. We may reduce to the case that \mathfrak{g} is semisimple. Restricting σ to $\mathfrak{g}^{\mathfrak{h}}$ we obtain that $\mathfrak{g}^{\mathfrak{h}} = (\mathfrak{h}^{\mathfrak{h}} = Z(\mathfrak{h})) \oplus \mathfrak{q}^{\mathfrak{h}}$. If \mathfrak{a} is a maximal commutative subalgebra of $\mathfrak{q}^{\mathfrak{h}}$, then $\exp(Z(\mathfrak{h}))$ applied to \mathfrak{a} generates the vector space $\mathfrak{q}^{\mathfrak{h}}$. Since the action of \mathfrak{h} is trivial on \mathfrak{a} , we must have that $\mathfrak{a} = \mathfrak{q}^{\mathfrak{h}}$, i.e., $\mathfrak{q}^{\mathfrak{h}}$ is abelian. Since \mathfrak{h} is reductive, we have an \mathfrak{h} -module decomposition $\mathfrak{q} = \mathfrak{q}^{\mathfrak{h}} \oplus \mathfrak{q}'$ for some \mathfrak{h} -module \mathfrak{q}' . Let \mathfrak{a} be a maximal abelian subspace of \mathfrak{q} containing $\mathfrak{q}^{\mathfrak{h}}$. Then $\mathfrak{a} = \mathfrak{q}^{\mathfrak{h}} \oplus \mathfrak{a}'$ where $\mathfrak{a}' \subset \mathfrak{q}'$. Again we have the fact that $\exp(\mathfrak{h}) \cdot \mathfrak{a}'$ has to generate \mathfrak{q}' . But then it follows that every element of \mathfrak{q}' commutes with $\mathfrak{q}^{\mathfrak{h}}$. Thus $\mathfrak{q}^{\mathfrak{h}}$ centralizes \mathfrak{q} . Since \mathfrak{h} obviously commutes with $\mathfrak{q}^{\mathfrak{h}}$, we have shown that $\mathfrak{q}^{\mathfrak{h}} \subset Z(\mathfrak{g})$. \square

7. STRATIFICATION AND WEYL GROUPS

We have the stratification of A_0 coming from the conjugacy class of the H -isotropy group. So a and b are in the same stratum if and only if H_a is conjugate to H_b . Another way to think of this is to map A_0 into $X//H$ and pull back the stratification of $X//H$ by isotropy type. We refine this stratification by taking connected components of the strata. Since the isotropy type stratification is locally finite, it is finite when restricted to A_0 .

We will show that if S is a stratum, then the translated tori Au , where A is maximal (σ, θ_u) -split and $u \in S$, are independent of u . Moreover, if $\bar{S}_1 \cap S_2 \neq \emptyset$ where the S_i are strata, then the tori for $u \in S_1$ are included in those for $u \in S_2$. Thus if one wants to find the maximal tori of Theorem 5.6, one needs to only consider those S_i which are minimal, i.e., closed. On the other hand, we show that to every stratum S there is associated at most one Au , and a subset of these tori gives a collection as in Theorem 5.6. Later we will see that we can determine the strata of A_0 using a Weyl group.

If $u \in A_0$, let $A_{0,u}$ denote $(A_0^{\text{conj}(H_u)})^0$. The stratum $X^{(H_u)}$ of X corresponding to H_u is $\{x \in X \mid H_{x_0} \text{ is conjugate to } H_u \text{ where } H * x_0 \subset \overline{H * x} \text{ is closed}\}$. Restricted to A_0 the stratum is just those points with isotropy group conjugate to H_u .

Lemma 7.1. *Let $u \in A_0$. Then $X^{(H_u)} \cap A_0$ is contained in a finite union $\cup_i A_{0,u_i} u_i$.*

Proof. The fixed point set $A_0^{H_u}$ for the $*$ -action is just $A_0^{\text{conj}(H_u)} u$. If a point $u' \in A_0$ is H -conjugate to a point of $A_0^{\text{conj}(H_u)} u$, then it is W_0^* -conjugate, so that we get a finite union covering $X^{(H_u)} \cap A_0$ as claimed. \square

Since only finitely many strata intersect A_0 we have

Corollary 7.2. *There is a finite W_0^* -stable collection $\{A_{0,u_i} u_i\}$ with the following properties.*

- (1) *For $i \neq j$, $A_{0,u_i} u_i \neq A_{0,u_j} u_j$.*
- (2) *If $u \in A_0$, then $u \in A_{0,u_i} u_i$ for some i where $H_{u_i} = H_u$.*

Now for a fixed i , let V_i denote the union of the $A_{0,u_j} u_j$ such that H_{u_i} is properly included in H_{u_j} . Then $T_i := A_{0,u_i} u_i \setminus V_i$ is Zariski open and dense in $A_{0,u_i} u_i$ and is the set of points of $A_{0,u_i} u_i$ whose isotropy group is H_{u_i} . Now define the strata of A_0 to be the connected components of the T_i .

Proposition 7.3. *Let $u \in A_0$ and let S be the stratum containing u . Then for $u' \in S$ we have $H_u = H_{u'}$, $P_u = P_{u'}$ and $P_u u = P_{u'} u'$.*

Proof. We have $u' = au$ for some $a \in A_{0,u}$. By definition, $H_u = H_{u'}$. Now the slice representation of H_u is its action on $\mathcal{S}_u = \mathfrak{g}^{(u)} \cap \mathfrak{q}$. Moreover, $\mathfrak{g}^{(u)} \cap \mathfrak{h}$ is the Lie algebra of H_u . Since P_u is determined by $\mathfrak{g}^{(u)}$, we see that P_u is determined by the slice representation at u . But slice representations are constant, up to isomorphism, along connected components of isotropy strata. By Lemma 6.3 applied to the action of σ on $\mathfrak{g}^{(u)}$ we obtain that $(\mathfrak{g}^{(u)})^{H_u} \cap \mathfrak{q}$ lies in the center of $\mathfrak{g}^{(u)}$. It follows that $A_{0,u}$ centralizes $\mathfrak{g}^{(u)}$ so that $\mathfrak{g}^{(u')} \supset \mathfrak{g}^{(u)}$. Thus $\mathfrak{g}^{(u')} = \mathfrak{g}^{(u)}$ and $P_u = P_{u'}$. Since $A_{0,u}$ centralizes $\mathfrak{g}^{(u)}$, it acts as translations on P_u . Thus $P_u u = P_{u'} u'$. \square

Corollary 7.4. *Let S be a stratum of A_0 . Then $\{Au : A \text{ is maximal } (\sigma, \theta_u)\text{-split}\}$ is independent of $u \in S$.*

Definition 7.5. We say that a maximal (σ, θ_u) -split and δ -stable torus A is *standard* if $A \cap U \subset A_0$.

Proposition 7.6. *Let A be maximal (σ, θ_u) -split where $u \in A_0$. Then there is an $h \in H_0$ such that $h * Au$ is standard.*

Proof. The torus A is maximal σ -split in $G^{(u)}$, and it follows from [ÖM80, Remark after Lemma 5] that we may conjugate A by an element of $((H_0)_u)^0$ such that it becomes δ -stable. So we may assume that A is δ -stable. Write $A = BC$ where B is a δ -split torus and C is a compact torus. By Theorem 3.1, the intersection of any orbit $H_0 * cu$ with Cu is finite, $c \in C$. Thus the dimension of the intersection of $H_0 * Cu$ with A_0 is the dimension of C . Hence there is a stratum S of A_0 of dimension at least $\dim C$ such that the orbit of an open subset of Cu intersects S . Now pick a cu in the open set. We can also assume that there is a $b \in B$ such that bcu is a principal point of Au . Thus we can reduce to the case that u lies in a stratum S of dimension at least $\dim C$ and that there is a principal point $x := bu$, $b = \exp(Z) \in B$ where $Z \in \mathcal{S}_u \cap \mathfrak{r}_0$.

From the proof of 5.6 we see that the Lie algebra of the unique (σ, θ_x) -split maximal torus through x is $\{Y \in \mathcal{S}_u \mid [Z, Y] = 0\}$, which is, of course, \mathfrak{a} . Now if $a \in A_0$ and $au \in S$, we know that $\theta_{au}(Z) = -Z$ and that $\theta_u(Z) = -Z$. Thus a centralizes Z so that $au \in Cu$ since S is connected. But $\dim S \geq \dim C$. Hence a neighborhood of the identity in C lies in A_0 . Hence $C \subset A_0$ so that A is standard. \square

Proposition 7.7. *Let S be a stratum of A_0 whose closure intersects the stratum T . Let $u \in S$ and let A be a maximally (σ, θ_u) -split torus in X . Then A is (σ, θ_v) -split for $v \in T$.*

Proof. By Corollary 7.4 we may assume that $v \in T$ is in the closure of S . Then clearly A is θ_v -split. \square

Corollary 7.8. *Let S_1, \dots, S_r be the closed strata of A_0 and consider a pair (A, u) where $u \in A_0$ and A is a maximal (σ, θ_u) -split torus. Then for some $v \in \cup_i S_i$, A is maximal (σ, θ_v) -split.*

If one only wants to find the maximal tori (appropriately split), then one only has to look at the minimal strata. One also only has to look at standard tori. However, this is not an algorithm. We give an algorithm after Proposition 7.16 below.

From Remark 5.2 we have an H_0 -equivariant projection $\pi: \mathcal{M} \rightarrow X_0$, $\exp(\xi)u \mapsto u$. Let π_0 denote the induced mapping from $\pi^{-1}(A_0) \rightarrow A_0$. From Theorem 4.6 we have an isomorphism $X_0/H_0 \simeq A_0/W_0^*$. Let $\rho: A_0 \rightarrow A_0/W_0^*$ be the quotient mapping. Then from our isomorphism and π we obtain a surjection $\gamma: \mathcal{M}/H_0 \rightarrow A_0/W_0^*$. We show that π_0 and γ are product bundles over appropriate strata.

Theorem 7.9. *Let S be a stratum of A_0 and let $u \in S$. Then, over S , π_0 is a product bundle $S \times \exp(S_u \cap \mathfrak{t}_0) \rightarrow S$ and, over $\rho(S)$, γ is a product bundle $\rho(S) \times \exp(\mathcal{S}_u \cap \mathfrak{t}_0)/(H_0)_u \rightarrow \rho(S)$.*

Proof. From the description of \mathcal{M} (Proposition 5.1) and Proposition 7.3 we see that $\pi_0^{-1}(S) \simeq S \times \exp(\mathcal{S}_u \cap \mathfrak{t}_0)$. Thus π_0 is a trivial bundle over S . Now let $\exp(Z_1)u_1, \exp(Z_2)u_2 \in \pi_0^{-1}(S)$ where $u_1, u_2 \in S$ and $Z_1, Z_2 \in \mathcal{S}_u \cap \mathfrak{t}_0$. Suppose that $h \in H_0$ and $h * \exp(Z_1)u_1 = \exp(Z_2)u_2$. Then $h * u_1 = u_2$, so there is an $h' \in N_0^*$ such that $h' * u_1 = u_2$. Note that, since W_0^* permutes the strata, $h' * S = S$. Then $(h')^{-1}h$ fixes u_1 , hence it fixes all the points of S . Let W_S^* denote the elements of W_0^* which send S into itself modulo the elements fixing S . Then W_S^* acts freely on S and $\rho(S) \simeq S/W_S^*$. Moreover, $\gamma^{-1}(\rho(S)) \simeq \rho(S) \times \exp(\mathcal{S}_u \cap \mathfrak{t}_0)/(H_0)_u$. \square

Now consider $u \in S$ and the group G_1 which is the fixed group of $\sigma\delta$ acting on $G^{(u)}$. Then G_1 is reductive with Cartan involution δ , and one easily sees that we have the Cartan decomposition $G_1 \simeq (H_0)_u \exp(\mathcal{S}_u \cap \mathfrak{t}_0)$. Let \mathfrak{t} be a maximal abelian subspace of $\mathcal{S}_u \cap \mathfrak{t}_0$ and let $W(S, \mathfrak{t})$ denote the associated Weyl group (the normalizer of \mathfrak{t} in $(H_0)_u$ divided by the corresponding centralizer). Then $\exp(\mathcal{S}_u \cap \mathfrak{t}_0)/(H_0)_u \simeq \exp(\mathfrak{t})/W(S, \mathfrak{t})$ (see, for example, [HelS01, Theorem 11.11]). We call $W(S, \mathfrak{t})$ the *Weyl group of S and \mathfrak{t}* . It is independent of $u \in S$ and the choice of \mathfrak{t} , up to isomorphism (see 6.1).

Corollary 7.10. *Let S be a stratum of A_0 , $u \in S$, and let \mathfrak{t} be a maximal toral subalgebra of $\mathcal{S}_u \cap \mathfrak{t}_0$. Then the fibers of γ above $\rho(S)$ are isomorphic to $\exp(\mathfrak{t})/W(S, \mathfrak{t})$.*

We now have the result that H -conjugacy is the same as H_0 -conjugacy for translated tori.

Theorem 7.11. *Let A_i be maximal (σ, θ_{u_i}) -split tori, $i = 1, 2$. Let $h \in H$ such that $h * A_1 u_1 = A_2 u_2$. Then there is an $h' \in H_0$ such that $h' * A_1 u_1 = A_2 u_2$.*

Proof. We can assume that the $A_i u_i$ are standard. Since $h * u_1 \in \mathcal{M}$, there is an $h_1 \in H_0$ such that $h * u_1 = h_1 * u_1$. Thus $h * u_1 \in A_2 u_2 \cap U$, hence $h * u_1 = a_2 u_2$ where $a_2 \in A_2 \cap U \subset A_0$. Now $\text{conj}(h)$ must send the maximal compact subgroup $A_1 \cap U$ of A_1 to $A_2 \cap U$, so that $h * (A_1 u_1 \cap U) = A_2 u_2 \cap U$. Write $h = \exp(Y)h_0$ where $h_0 \in H_0$ and $Y \in \mathfrak{h} \cap \mathfrak{t}_0$. Then by uniqueness of the Cartan decomposition one obtains that $h * = h_0 *$ on $A_1 u_1 \cap U$. Thus we may assume that $h *$ is the identity on $A_1 u_1 \cap U$. Then the projections of \mathfrak{a}_1 and \mathfrak{a}_2 to $\mathfrak{q} \cap \mathfrak{t}_0$ are maximally σ -split subalgebras commuting with $\mathfrak{a}_1 \cap \mathfrak{a}_0$. By Proposition 6.1 there is an $h' \in ((H_0)_{u_1})^0$ centralizing $A_1 \cap A_0$ such that $\text{Ad}(h')\mathfrak{a}_1 = \mathfrak{a}_2$. \square

Remark 7.12. In the last line of the proof above, $h' \in (H_0)_{u_1}$ already implies that h' fixes all of the stratum S containing u_1 , hence that it centralizes $A_1 \cap A_0$ (which is generated by Su_1^{-1}).

From Corollary 4.5 we obtain

Corollary 7.13. *Let the $A_i u_i$ be standard maximal (σ, θ_{u_i}) -split tori, $i = 1, 2$. Then $A_1 u_1$ and $A_2 u_2$ are equivalent if and only if there is a $w \in W_0^*$ such that $w * (A_1 u_1 \cap A_0) = A_2 u_2 \cap A_0$.*

Corollary 7.14. *Let A be maximal (σ, θ_u) -split where $u \in A_0$. Then $W_H^*(Au)$ is finite*

Proof. We may assume that Au is standard. Let $h \in W_{H_0}^*(Au) = W_H^*(Au)$. Modifying h by an element of W_0^* we may assume that h is the identity on $Au \cap A_0$. Then the projection \mathfrak{t} of \mathfrak{a} to $\mathcal{S}_u \cap \mathfrak{r}_0$ is maximal abelian and h induces an element of $W(S, \mathfrak{t})$ where S is the stratum containing u . Thus $W_H^*(Au)$ is finite. \square

Corollary 7.15. *Let the $A_i u_i$ be as in Theorem 5.6. Then for each i the mapping from the principal points of $A_i u_i$ to $X//H$ is the quotient by a free action of $W_{H_0}^*(A_i u_i)$.*

Proposition 7.16. *Let Au be a standard maximal (σ, θ_u) -split torus. Let B denote $A \cap A_0$. Let S be a stratum of A_0 such that $Bu \cap S$ has the same dimension as S . Then, up to equivalence, Au is determined by S .*

Proof. We may assume that $u \in S$. Now $\mathfrak{a} \cap \mathfrak{r}_0$ is maximal σ -split in $\mathcal{S}_u \cap \mathfrak{r}_0$. By Corollary 6.2 and Remark 7.12, $\mathfrak{a} \cap \mathfrak{r}_0$ is unique, up to the action of $(H_0)_u$. Since $A \cap A_0$ is generated by Su^{-1} , A is determined by S , up to equivalence. \square

Now we give an algorithm to find a minimal collection of standard tori $A_i u_i$ whose images cover $X//H$. Let S_1, \dots, S_r be strata of A_0 such that the strata of A_0/W_0^* are the $\rho(S_i)$. Let S be one of the S_i and let $u \in S$. Let \mathfrak{t} be a maximal toral subalgebra in $\mathcal{S}_u \cap \mathfrak{r}_0$. If $\dim \mathfrak{t} + \dim S = \dim X//H$, then let C be the connected subtorus of A_0 generated by Su^{-1} and let B denote $\exp(\mathfrak{t})$. Then $A := BC$ is a maximal (σ, θ_u) -split torus and we add Au to our collection. If $\dim \mathfrak{t} + \dim S < \dim X//H$ do not add anything. By Proposition 7.16 we obtain a set $\{A_i u_i = B_i C_i u_i\}$ whose images cover $X//H$. Now it is only a matter of removing one of any pair of tori $A_i u_i$ and $A_j u_j$ whenever $C_i u_i$ and $C_j u_j$ are carried one to the other by an element of W_0^* . Thus we obtain a minimal collection as in Theorem 5.6.

Remark 7.17. Let S_1, \dots, S_p be the open strata of A_0 and let A be a maximal standard (σ, θ) -split torus containing A_0 . Then, up to equivalence, $A_1 u_1 = \dots = A_p u_p = A$.

Example 7.18. We revisit Example 5.9. Let $u = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in A_0 = \text{SO}(2)$. Since $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in H$ sends u to $\begin{pmatrix} \lambda^2 a & b \\ -b & \lambda^{-2} a \end{pmatrix}$, the isotropy group of H at u is $\pm I$ for $u \neq \pm v$ and H for $u = \pm v$. Thus the strata of A_0 are the two points $\pm v$ and the two connected components of $A_0 \setminus \{\pm v\}$. Since W_0^* is trivial, these are also the strata of A_0/W_0^* . The maximal (σ, θ) -split torus A_0 comes from the open strata and the translated tori $\{(\begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a^2 - b^2 = 1, a > 0\}(\pm v)$ correspond to the two closed strata.

8. WHEN $K \neq G^\theta$

Let K' be an open subgroup of $K = G^\theta$ and let $\beta: X' = G/K' \rightarrow X$ be the corresponding cover. Let \mathcal{M}' denote $\beta^{-1}(\mathcal{M})$. We show that \mathcal{M}' can serve as a Kempf-Ness set for the H -action on X' .

Lemma 8.1. *Let $x \in X'$. Then the orbit Hx is closed if and only if it intersects \mathcal{M}' .*

Proof. If $x \in \mathcal{M}'$, then the H -orbits in the closed set $\beta^{-1}(H\beta(x))$ are open and closed, hence Hx is closed. Conversely, if Hx is closed, then $H\beta(x)$ intersects \mathcal{M} , so that Hx intersects \mathcal{M}' . \square

Lemma 8.2. *Let $x \in X'$. Then $\overline{Hx} \cap \mathcal{M}'$ is a single H_0 -orbit.*

Proof. Since H_0 acts transitively on the connected components of H , we can reduce to the case that H is connected. For $y \in X$, let $f(y)$ denote the norm squared of $\mu(y)$ using the invariant inner product B of Lemma 2.12, and consider the flow ψ_t on X which is the negative of the gradient of f . (We can use the inner product on X induced by the Kähler form on $U_{\mathbb{C}}$ §2.13.) Then ψ is a flow along H -orbits and it is H_0 -equivariant. Moreover, since all our data is real analytic, the flow extends to $+\infty$ to give a deformation retraction of X onto \mathcal{M} (see [Sch88] and [HSt07]). Now the flow lifts to X' to give a deformation retraction ψ'_t of X' onto \mathcal{M}' . If $x \in X'$, then $\psi'_\infty(\overline{Hx})$ is connected. We know that the lemma holds for $\beta(x)$ in place of x , so that $\overline{Hx} \cap \mathcal{M}'$ is a finite union of H_0 -orbits which cover $\overline{H\beta(x)} \cap \mathcal{M}$. But connectedness now implies that there is only one H_0 -orbit. \square

Corollary 8.3. *The analogues of Theorems 2.16 and 2.17 hold for X' and \mathcal{M}' and we have a quotient $X'//H$.*

Let X'_0 denote G_0/K'_0 where $K'_0 = K' \cap U$. Then $\beta: X'_0 \rightarrow X_0$ is a cover of the same degree as $X' \rightarrow X$. Then one easily shows

Corollary 8.4. $\mathcal{M}' = \{\exp(Z)v \mid v \in X'_0 \text{ and } Z \in \mathcal{S}_{\beta(v)} \cap \mathfrak{r}_0\}$.

Remark 8.5. Let $v, v' \in A_0K'_0$, let $Z \in \mathcal{S}_{\beta(v)} \cap \mathfrak{r}_0$ and let $Z' \in \mathcal{S}_{\beta(v')} \cap \mathfrak{r}_0$. Then $\exp(Z)vK' = \exp(Z')v'K'$ if and only if $Z = Z'$ and $v = v'$.

Let S be a stratum of A_0 , and let S' denote $\beta^{-1}(S) \cap A_0K'_0$. Then $S' = \cup_{j=1}^r S'_j$ where the S'_j are connected and are covering spaces of S . Let $u \in S$ and $h \in (H_u)_v$, $v \in S'_j$ for some j . Then h preserves S'_j and if h does not act trivially on S'_j , then, since $S'_j \rightarrow S$ is an $(H_u)_v$ -equivariant cover, it follows that h acts nontrivially on S , a contradiction. Thus $(H_u)_v$ acts trivially on S'_j and is independent of $v \in S'_j$, so that we can use $\cup_S \cup_j S'_j$ as strata for $A_0K'_0$. Let A be $(\sigma, \theta_{\beta(v)})$ -split and δ -stable where $v \in A_0K'_0$. We say that Av (or A) is *standard* if $A \cap U \subset A_0$. Then as in Proposition 7.6 one shows that for every maximal $(\sigma, \beta(v))$ -split torus A , $v \in A_0K'_0$, there is an $h \in H_0$ such that $hAv = hAh^{-1}hv$ is standard.

Let $\pi': \mathcal{M}' \rightarrow X'_0$ be the canonical map, and let $\pi'_0: (\pi')^{-1}(A_0K'_0) \rightarrow A_0K'_0$ be the induced mapping. Let γ' be the canonical mapping of $\mathcal{M}' \rightarrow X'_0/H_0 \simeq (A_0K'_0)/W'_0$ and let $\rho': A_0K'_0 \rightarrow A_0K'_0/W'_0$ be the quotient mapping. Then we have the analogue of Theorem 7.9:

Theorem 8.6. *Let S' be a stratum of $A_0K'_0$ and let $v \in S'$. Then, over S' , π'_0 is a product bundle $S' \times \exp(\mathcal{S}_{\beta(v)} \cap \mathfrak{r}_0) \rightarrow S'$ and, over $\rho'(S')$, γ' is a product bundle $\rho'(S') \times \exp(\mathcal{S}_{\beta(v)} \cap \mathfrak{r}_0)/(H_0)_v \rightarrow \rho'(S')$.*

There is also clearly the analogue of Corollary 7.10, whose statement we omit. For A maximal $(\sigma, \beta(v))$ -split, let $W'_H(Av)$ denote the group of automorphisms of Av coming from elements of H , and define $W'_{H_0}(Av)$ similarly. Then the obvious analogues of Theorem 7.11 and Corollary 7.13 hold and the group $W'_H(Av) = W'_{H_0}(Av)$ is finite.

If one wants to find translated tori Av which minimally cover $X'//H$, then one can proceed as before (discussion after Proposition 7.16). Here is another method. Let Au be one of the maximal tori occurring in Theorem 5.6. We can assume that it is standard. It is covered by tori Av_j where $\beta(v_j) = u$, $v_j \in A_0K'$. Let $W'_{0,u}$ denote the subgroup of W'_0 whose image in W_0^* fixes $(A \cap A_0)u$. Then $W'_{0,u}$ acts on $\beta^{-1}(u) \cap A_0K'$ and one chooses a maximal subcollection of the Av_j where the v_j are on disjoint $W'_{0,u}$ orbits. Then starting with a minimal collection $A_i u_i$ as in Theorem 5.6 one arrives at a minimal collection $A_i v_{ij}$ which surjects onto $\mathcal{M}'/H_0 \simeq X'//H$.

9. STRATIFICATION VIA A WEYL GROUP

Now we would like to compare our stratification of A_0 with that given by a Weyl group. It turns out that $W_H^*(A_0) = W_0^*$ is not large enough for our purposes (see Example 10.1 below).

Thus we have to consider the action of the Weyl group $W := W_{H_{\mathbb{C}}}^*(A_0)$ where $H_{\mathbb{C}}$ is the Zariski closure of H in $U_{\mathbb{C}}$. Since $e \in \mathcal{M}$, the orbits $H * e$ and $H_{\mathbb{C}} * e$ are closed and the argument of Remark 4.2 shows that W is finite.

Proposition 9.1. *Let $\{S_i\}$ be the set of connected H -isotropy type strata of A_0 and let $\{T_j\}$ be the connected strata for the action of $W = W_{H_{\mathbb{C}}}^*(A_0)$. Then $\{T_j\}$ is a refinement of $\{S_i\}$, i.e., every S_i is a union of some of the T_j .*

Proof. Let $u \in A_0$. We have to show that the stratum of A_0 containing u for the H -isotropy stratification contains the corresponding stratum for the W -isotropy stratification near the point u . We can consider everything in the transversal at the point u , so we replace G by $G^{(u)}$, $U_{\mathbb{C}}$ by $U_{\mathbb{C}}^{(u)}$ and $H_{\mathbb{C}}$ by $H_{\mathbb{C}} \cap U_{\mathbb{C}}^{(u)}$. Thus we may assume that $u = e$ and that $\sigma = \theta$. Now the complexification $A_{\mathbb{C}}$ of A is a maximal σ -split torus in $U_{\mathbb{C}}$. The subgroup W_e of W is generated by elements of $H_{\mathbb{C}}$ which preserve A_0 under conjugation. Near e , the stratum of H is B_1 where $B_1 := \{b \in A_0 \mid H \cdot b = b\}$ and the stratum of W_e is B_2 where $B_2 := \{b \in A_0 \mid W_e \cdot b = b\}$. We need to show that $B_2^0 \subset B_1^0$.

Consider B_2 and let $\alpha: A_0 \rightarrow S^1$ be a root of the A_0 -action on \mathfrak{g} . Let $A_0 \subset A$ where A is maximal σ -split in G . We show that there is an element of W_e which sends α to α^{-1} , as follows. The root α is the restriction to A_0 of a root of A and a root of the complexification $A_{\mathbb{C}}$, which we will also call α . Let $(A_{\mathbb{C}})_{\alpha}$ be the kernel of α and consider the centralizer of $(A_{\mathbb{C}})_{\alpha}$ in $U_{\mathbb{C}}$ divided by $(A_{\mathbb{C}})_{\alpha}$. Call this group C . Then C has maximal σ -split torus $A_{\mathbb{C}}/(A_{\mathbb{C}})_{\alpha}$, and the roots of this torus acting on \mathfrak{c} are multiples of α . Then there is a $c \in (C \cap H_{\mathbb{C}})^0$ which acts as inverse on $A_{\mathbb{C}}/(A_{\mathbb{C}})_{\alpha} \simeq \mathbb{C}^*$ [Ric82]. Thus c gives us an element of W_e which fixes $(A_0)_{\alpha}$ and acts as inverse on $A_0/(A_0)_{\alpha}$. It follows that $B_2^0 \subset \text{Ker } \alpha$. Since α is an arbitrary root of A_0 , we find that B_2^0 centralizes $\mathfrak{u}_{\mathbb{C}}$, hence B_2^0 is fixed by $H_{\mathbb{C}}^0$. If $h \in H$, then h sends A to a maximal σ -split subtorus $\text{conj}(h)A$ of G . Since A is maximally compact, by Corollary 6.2 there is an element of $(H_0)^0$ which when composed with h stabilizes A . Thus, modulo H^0 (which fixes B_2^0), we may assume that $h \in N_H(A)$. Thus h preserves A_0 and it acts on B_2^0 as an element of W_e , i.e., trivially. Thus $B_2^0 \subset B_1^0$. \square

If K' is an open subgroup of K , then define $W' := W'_{H_{\mathbb{C}}}(A_0 K'_0)$ as the subgroup of $H_{\mathbb{C}}$ stabilizing $A_0 K'_0$ divided by the subgroup centralizing $A_0 K'_0$. Then W' is finite since $W' \rightarrow W$ has finite kernel and W is finite.

Corollary 9.2. *Let $\{S'_i\}$ be the set of connected H -isotropy type strata of $A_0 K'_0$ and let $\{T'_j\}$ be the connected strata for the action of W' . Then $\{T'_j\}$ is a refinement of $\{S'_i\}$, i.e., every S'_i is a union of some of the T'_j .*

Proof. Let $v \in A_0 K'_0$ and set $u = \beta(v)$. Set $C_1 = (A_0 K'_0)^{H_v}$ and $C_2 = (A_0 K'_0)^{W'_v}$. We need to show that $C_2^0 \subset C_1^0$. By Proposition 9.1 we have that $B_2^0 \subset B_1^0$ where $B_2 = A_0^{W_u}$ and $B_1 = A_0^{H_u}$. Note that $[H_u : H_v]$ is finite and that $C_1^0 = (B_1^{H_v})^0 K'_0$. We have the analogous result for C_2^0 . If $h \in (H_u)^0$, then h fixes v and B_2^0 . Clearly every element of $H_v/(H_u)^0$ is represented by an element normalizing A_0 . Thus H_v acts trivially on $((B_2 K'_0)^{W'_v})^0 = C_2^0$ so that $C_2^0 \subset C_1^0$. \square

10. SOME EXAMPLES

Example 10.1. . We reexamine Example 5.9, 7.18 using the Weyl group. The group W_0^* is trivial, but $W_{H_{\mathbb{C}}}^*(A_0)$ is generated by the element $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ which acts on A_0 by sending $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ to $\begin{pmatrix} -a & b \\ b & -a \end{pmatrix}$. The fixed points of this action are $\{\pm v\}$, hence we obtain again the isotropy type stratification of H .

Example 10.2. Suppose that $A_0 = \{e\}$. By 6.2 there is only one $(H_0)^0$ -conjugacy class of maximal (σ, θ) -split tori in G . Thus $\mathcal{M} = H_0 * A$ where A is maximal (σ, θ) -split. The argument

of Proposition 4.3 shows that $\mathcal{M}/H_0 \simeq A/W_{H_0}^*(A) \simeq X//H$. Now suppose that σ and θ commute. Then for $h \in N_{H_0}^*(A)$, $\beta(h) \in A$ is an element of order 2, hence it is trivial. Thus $W_{H_0}^*(A) = W_{M_0}(A)$ is an ordinary Weyl group where $M_0 = H_0 \cap K_0$. This agrees with [HelS01, 11.11] where it is assumed that θ is a Cartan involution.

Now suppose that every maximal (σ, θ) -split torus in G is compact. Then again there is only one $(H_0)^0$ -conjugacy class of maximal (σ, θ) -split tori and we have $X//H \simeq A/W_{H_0}^*(A)$ where A is maximal (σ, θ) -split. This agrees with the case G compact §4.

Example 10.3. Here is a case where A_0 is of dimension two and the group W_0^* is large. Let $U_{\mathbb{C}} = \mathrm{SL}(8, \mathbb{C})$, $U = \mathrm{SU}(8, \mathbb{C})$ and $G = \mathrm{SL}(8, \mathbb{R})$. Let δ be the associated Cartan involution $g \mapsto {}^t\bar{g}^{-1}$, $g \in G$. Then $G_0 = \mathrm{SO}(8, \mathbb{R})$. The Lie algebra \mathfrak{t}_0 consists of the symmetric real matrices. Let θ be δ followed by conjugation with $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and let σ be conjugation with $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ where I is 4×4 . Then σ , θ and δ commute. The group H is the set of real matrices $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ such that $\det AB = 1$, and being in H_0 adds the condition that A and B are orthogonal. The group K is a copy of $\mathrm{Sp}(8, \mathbb{R})$. The dimension of the quotient of G by H and K is the dimension of the quotient of $\mathfrak{p} \cap \mathfrak{q}$ by the action of $M = H \cap K$. Now M consists of matrices $\begin{pmatrix} g & 0 \\ 0 & {}^tg^{-1} \end{pmatrix}$ where $g \in \mathrm{GL}(4, \mathbb{R})$ and $\mathfrak{p} \cap \mathfrak{q} = \{ \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \mid A \text{ and } B \text{ are skew symmetric} \}$. The representation of $\mathrm{GL}(4, \mathbb{R})$ is isomorphic to that on $\wedge^2 \mathbb{R}^4 \oplus \wedge^2 (\mathbb{R}^4)^*$ which gives us that the quotient has dimension 2.

Let J denote $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then

$$A_0 = \left\{ \begin{pmatrix} aI & 0 & bJ & 0 \\ 0 & a'I & 0 & b'J \\ bJ & 0 & aI & 0 \\ 0 & b'J & 0 & a'I \end{pmatrix} \mid a^2 + b^2 = 1, (a')^2 + (b')^2 = 1 \right\}$$

is a maximal (σ, θ) -split torus (since its dimension is 2) which is compact. Let $a(\eta_1, \eta_2)$ denote the element of A_0 as above with $a + ib = e^{i\eta_1}$ and $a' + ib' = e^{i\eta_2}$. Now we compute W_0^* . If $\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} \in H_0$, then it acts on an element $a \in A_0$ sending it to $\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} a \begin{pmatrix} h^t & 0 \\ 0 & g^t \end{pmatrix}$. First suppose that $g = \begin{pmatrix} \alpha & 0 \\ 0 & I \end{pmatrix}$ and $h = \begin{pmatrix} \alpha & 0 \\ 0 & I \end{pmatrix}$ where $\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}$ sends $a(\eta_1, \eta_2)$ to $a(-\eta_1, \eta_2)$. If we have $g = \begin{pmatrix} J & 0 \\ 0 & I \end{pmatrix}$ and $h = \begin{pmatrix} -J & 0 \\ 0 & I \end{pmatrix}$, then $a(\eta_1, \eta_2)$ is sent to $a(-\eta_1 + \pi, \eta_2)$. Thus W_0^* contains the translation $a(\eta_1, \eta_2) \mapsto a(\eta_1 + \pi, \eta_2)$. Of course we have elements of W_0^* which do similar things to η_2 while leaving η_1 fixed. Finally, let $g = h = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. Then the action of this element interchanges η_1 and η_2 . Thus we see that W_0^* contains the semidirect product $W(\mathbf{B}_2) \ltimes (\mathbb{Z}/2\mathbb{Z})^2$ where $W(\mathbf{B}_2)$ is the Weyl group of type \mathbf{B}_2 and $(\mathbb{Z}/2\mathbb{Z})^2$ consists of the pure translations in W_0^* . We claim that $W_{H_{\mathbb{C}}}^*(A_0)$ is no bigger than this. Now the set of pure translations $(\mathbb{Z}/2\mathbb{Z})^2$ is as large as possible (the pure translations have to be of order 2 [HelS01, 1.9]). Thus the remaining elements of $W_{H_{\mathbb{C}}}^*(A_0)$ fix $e \in A_0$, i.e., they are fixed by θ . Moreover, the Weyl group has representatives which are in U , so that we are reduced to calculating $W_{M \cap U}(A_0)$. Now $M \cap U = \{ \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \mid g \in \mathrm{U}(4, \mathbb{C}), \det g = \pm 1 \}$. A matrix calculation now shows that one cannot obtain elements of the Weyl group of A_0 that we have not already seen. Thus $W_{H_{\mathbb{C}}}^*(A_0) = W_0^* \simeq W(\mathbf{B}_2) \ltimes (\mathbb{Z}/2\mathbb{Z})^2$.

Now a fundamental domain D for the action of W_0^* on A_0 consists of the elements $a(\eta_1, \eta_2)$ where $0 \leq \eta_1 \leq \eta_2 \leq \pi/2$. Since $W_{H_{\mathbb{C}}}^*(A_0) = W_0^*$, the strata for the action of W_0^* and H are the same. We now list the various strata S occurring in D .

Case 1: $S = \{a(\eta_1, \eta_2) \mid 0 < \eta_1 < \eta_2 < \pi/2\}$ is the only two-dimensional stratum. The preimage of S in the quotient $X//H$ is just a copy of S .

Case 2: $S = \{a(\eta_1, \eta_2) \mid 0 < \eta_1 = \eta_2 < \pi/2\}$. Here the preimages of points of S are one-dimensional. Let $a \in S$. Then $(H_0)_a = \{ \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \mid g \text{ is orthogonal and } gJ_2g^{-1} = J_2 \}$ where

$J_2 = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}$ gives a complex structure on \mathbb{R}^4 . Thus $(H_0)_a \simeq \mathrm{U}(2, \mathbb{C})$. This group acts on $\mathcal{S}_a \cap \mathfrak{t}_0$ which consists of matrices $\begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix}$ where C is skew symmetric and $J_2 C = -C J_2$. This set of matrices is a complex vector space of dimension 2, and we have the standard action of $\mathrm{U}(2, \mathbb{C})$ on \mathbb{C}^2 . The Lie algebra \mathfrak{t} of a maximal torus in $\exp(\mathcal{S}_a \cap \mathfrak{t}_0)$ is generated by $\begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix}$ where $C = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}$ and $\alpha = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. Note that \mathfrak{t} is stable under the action of $(W_0^*)_a \simeq \mathbb{Z}/2\mathbb{Z}$ (generated by the element reversing the order of η_1 and η_2), and the action is multiplication by -1 . We have $W(S, \mathfrak{t}) \simeq \mathbb{Z}/2\mathbb{Z}$, and $\exp(\mathcal{S}_a \cap \mathfrak{t}_0)/(H_0)_a \simeq \mathbb{R}^{>0}/(\mathbb{Z}/2\mathbb{Z})$ where the Weyl group action sends $r \in \mathbb{R}^{>0}$ to $1/r$. Thus the preimage of S is isomorphic to $S \times \{r \in \mathbb{R} \mid r \geq 1\}$.

Case 3: $S = \{a(0, \eta_2) \mid 0 < \eta_2 < \pi/2\}$ or $S = \{\eta_1, \pi/2\} \mid 0 < \eta_1 < \pi/2\}$. As above, a maximal abelian subspace $\mathfrak{t} \subset \mathcal{S}_a \cap \mathfrak{t}_0$ has dimension 1 and $W(S, \mathfrak{t}) \simeq \mathbb{Z}/2\mathbb{Z}$. The fiber above any point $a \in S$ is isomorphic to $\{r \in \mathbb{R} \mid 1 \leq r\}$.

Case 4: $S = \{e = a(0, 0)\}$. From our calculations above we have that $(H_0)_e = H_0 \cap K_0 = \{\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \mid g \text{ is orthogonal}\}$ and that $\mathcal{S}_e \cap \mathfrak{t}_0 = \mathfrak{p} \cap \mathfrak{q} \cap \mathfrak{t}_0 = \{\begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix} \mid C \text{ is skew symmetric}\}$. Thus we have the adjoint representation of $\mathrm{O}(4, \mathbb{R})$. We choose as maximal toral subalgebra \mathfrak{t} the set of matrices

$$\begin{pmatrix} 0 & 0 & t_1 J & 0 \\ 0 & 0 & 0 & t_2 J \\ -t_1 J & 0 & 0 & 0 \\ 0 & -t_2 J & 0 & 0 \end{pmatrix}.$$

Then $(W_0^*)_e \simeq W(\mathcal{B}_2) \simeq W(S, \mathfrak{t})$ acts on \mathfrak{t} in the standard way. Exponentiating we get pairs of positive real numbers r_1 and r_2 whose normal form under $W(\mathcal{B}_2)$ consists of $\{(r_1, r_2) \mid 1 \leq r_1 \leq r_2\}$. This is the preimage of S .

Case 5: $S = \{a := a(\pi/2, \pi/2)\}$. Here $(H_0)_a$ consists of the matrices $\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}$ such that g and h are orthogonal and commute with J_2 . Thus our group is isomorphic to $\mathrm{U}(2, \mathbb{C}) \times \mathrm{U}(2, \mathbb{C})$. The vector space $\mathcal{S}_a \cap \mathfrak{t}_0$ consists of matrices $\begin{pmatrix} 0 & C \\ C^t & 0 \end{pmatrix}$ where C anticommutes with J_2 . This is a complex vector space of dimension 4, and $\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}$ acts sending C to gCh^{-1} . Thus we have the obvious action of $\mathrm{U}(2, \mathbb{C}) \times \mathrm{U}(2, \mathbb{C})$ on $\mathbb{C}^2 \otimes_{\mathbb{C}} \mathbb{C}^2$. As maximal toral subalgebra we can choose $\mathfrak{t} = \{\begin{pmatrix} 0 & t_1 \alpha \\ t_2 \alpha & 0 \end{pmatrix}\}$ where $\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $t_1, t_2 \in \mathbb{R}$. Here $(W_0^*)_a$ once again acts on \mathfrak{t} , but not faithfully. The image only contains the element interchanging $t_1 \alpha$ and $t_2 \alpha$. One easily calculates that $W(S, \mathfrak{t}) \simeq W(\mathcal{B}_2)$. Hence $\exp(\mathcal{S}_a \cap \mathfrak{t}_0)/(H_0)_a \simeq \{(r_1, r_2) \in \mathbb{R}^2 \mid 1 \leq r_1 \leq r_2\}$. This is the same fiber that we saw in Case 4.

Case 6: $S = \{a := a(0, \pi/2)\}$. Here the story is slightly different. Again $(W_0^*)_a$ acts on an appropriately chosen maximal abelian subspace $\mathfrak{t} \subset \mathcal{S}_a \cap \mathfrak{t}_0$ where $\mathfrak{t} \simeq \mathbb{R}^2$. The image of $(W_0^*)_a \simeq (\mathbb{Z}/2\mathbb{Z})^2$ in $W(S, \mathfrak{t})$ acts as a sign change on one of the coordinates while $W(S, \mathfrak{t}) \simeq (\mathbb{Z}/2\mathbb{Z})^2$ acts by sign changes of both coordinates. Thus the fiber above a is naturally isomorphic to $\{(r, s) \in \mathbb{R}^2 \mid r, s \geq 1\}$.

REFERENCES

- [HeiS07] P. Heinzner and G. Schwarz, *Cartan decomposition of the moment map*, Math. Ann. **337** (2007), 197–232.
- [HSt07] P. Heinzner and H. Stötzl, *Semistable points with respect to real forms*, Math. Ann. **338** (2007), 1–9.
- [Hel78] S. Helgason, *Differential geometry, Lie groups and symmetric spaces*, Pure and Applied mathematics, vol. XII, Academic Press, New York, 1978.
- [HelS01] A. G. Helminck and G. W. Schwarz, *Orbits and invariants associated with a pair of commuting involutions*, Duke Math. J. **106** (2001), no. 2, 237–279.
- [Mie07] C. Miebach, *Matsuki's double coset decomposition via gradient maps*, J. of Lie Theory **18**, (2008), 555–580.

- [Mat97] T. Matsuki, *Double coset decompositions of reductive Lie groups arising from two involutions*, J. Alg. **197** (1997), 49–91.
- [ŌM80] T. Ōshima and T. Matsuki, *Orbits on affine symmetric spaces under the action of the isotropy subgroups*, J. Math. Soc. Japan **32** (1980), no. 2, 399–414.
- [Ric82] R. W. Richardson, *Orbits, invariants and representations associated to involutions of reductive groups*, Invent. Math. **66** (1982), 287–312.
- [Sch88] G. W. Schwarz, *The topology of algebraic quotients*, Topological methods in algebraic transformation groups (New Brunswick, NJ, 1988), Progr. Math. **80** (1989), Birkhäuser, Boston, 135–151.

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